

A linear potential in a light cone QCD inspired model

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Abstract. The general equation from previous work is specialized to a linear potential $V(r) = -a + Fr$ acting in the space of spherically symmetric S wave functions. The fine and hyperfine interaction creates then a $\frac{1}{r}$ -dependence in the effective potential energy equation and a position dependent mass $\tilde{m}(r)$ in the effective kinetic energy of the associated Schrödinger equation. The results are compared with the available experimental and theoretical spectral data on the π and ρ . Solving the eigenvalue problem within the analytically tractable Airy-function approach induces a certain amount of arbitrariness (fudge factors). Despite of this, the agreement with experimental data is good and partially better than other calculations, including Godfrey and Isgur [9] and Baldicchi and Prosperi [10]. The short comings of the present model can be removed easily in more elaborate work.

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1 The S-state Hamiltonian

For spherically symmetric S states the previously derived Hamiltonian reduces in Fourier approximation to [1,2,3]

$$\begin{aligned} H &= \frac{\mathbf{p}^2}{2m_r} + V + V_{hf} + V_K + V_D, \\ V_{hf} &= \frac{\sigma_1 \sigma_2}{6m_1 m_2} \nabla^2 V, \\ V_K &= \frac{V}{m_1 m_2} \mathbf{p}^2, \\ V_D &= -\left[\frac{V}{16m_1 m_2} \mathbf{p}^2 + \frac{\nabla^2 V}{4m_1 m_2} \right] \left(\frac{m_1}{m_2} + \frac{m_2}{m_1} \right). \end{aligned} \quad (1)$$

There are no more interactions than the central potential, the hyperfine, the kinetic, and the Darwin interaction, but also no less. For s-states the total spin squared is a good quantum number $\mathbf{S}^2 = [(\sigma_1 + \sigma_2)/2]^2 = S(S+1)$, thus

$$\sigma_1 \sigma_2 = 2S(S+1) - 3 = \begin{cases} +1, & \text{for } S = 1, \text{ triplet,} \\ -3, & \text{for } S = 0, \text{ singlet.} \end{cases} \quad (2)$$

Because it is shorter, $\sigma_1 \sigma_2$ is kept explicit in the equations as an abbreviation for Eq.(2). Choosing a linear potential,

$$V(r) = -a + Fr, \quad (3)$$

with the force parameter F , often called string tension σ , the Hamiltonian (1) becomes a non-local Schrödinger equation with a $\frac{1}{r}$ -potential

$$\begin{aligned} H &= \left[\frac{1}{2m_r} + \frac{V(r)}{m_1 m_2} \right. \\ &\quad + \left[\frac{Fr}{3m_1 m_2} \sigma_1 \sigma_2 - \frac{F}{2m_1 m_2} \left(\frac{m_1}{m_2} + \frac{m_2}{m_1} \right) \right] \mathbf{p}^2 \\ &\quad \left. + \left[\frac{V(r)}{16m_1 m_2} \left(\frac{m_1}{m_2} + \frac{m_2}{m_1} \right) - a \right] \right] \frac{1}{r}, \end{aligned}$$

since

$$\nabla^2 V(r) = \frac{1}{r} \frac{d^2}{dr^2} r V(r) = 2 \frac{F}{r}. \quad (4)$$

Shaping notation, the Hamiltonian is written as

$$H = \frac{\mathbf{p}^2}{2\tilde{m}_r(r)} + Fr - a - \frac{\beta}{r} \left(\frac{1}{2} \left[\frac{m_1}{m_2} + \frac{m_2}{m_1} \right] - \frac{\sigma_1 \sigma_2}{3} \right). \quad (5)$$

The dimensionless ‘coupling constant’ is

$$\beta = \frac{F}{m_2 m_1}. \quad (6)$$

The spin-averaged potential energy for equal masses,

$$V_{av} \equiv \frac{3V_t + V_s}{4} = Fr - a - \frac{\beta}{r}, \quad (7)$$

has an attractive Coulomb potential. It has its origin in the Darwin term. The non locality of the Hamiltonian resides in the position dependent mass

$$\frac{m_r}{\tilde{m}_r(r)} = 1 + \frac{V(r)}{8(m_1 + m_2)} \left[16 - \frac{m_1}{m_2} - \frac{m_2}{m_1} \right]. \quad (8)$$

To solve this Hamiltonian, one must go on a computer.

The Hamiltonian in Eq.(5) looks like a conventional instant form Hamiltonian as obtained by quantizing the system at equal usual time. But it must be emphasized that it continues to be a genuine front form or light cone Hamiltonian [4], derived from the latter by a series of exact unitary transformations [1,2].

2 The model Hamiltonian and its parameters

In this first round, I try to avoid to go on the computer as far as possible, by the following reason. The parameters in

Table 1. Model parameters in GeV. Note: $[f^*] = 1$.

f_r^*	f_i^*	$m_{d,u}$	m_s	m_c	m_b	a	$10F$
2	1	0.4259	0.5553	1.8152	5.2505	1.1317	2.1454

the theory must be determined from experiment, and this turns out as a non trivial, strongly non linear problem. In order to get a first and rough estimate, the Hamiltonian is simplified here until it has a form which is amenable to analytical solution. Therefore, all in-tractable terms in the above will be replaced here by mean values and related to the experimentally accessible mean square radius $\langle r^2 \rangle$ [5].

In effect, the substitution

$$\tilde{m}_r(r) \Rightarrow \tilde{m}_r, \quad (9)$$

is the only true assumption in the present model. I consider thus the model Hamiltonian,

$$H = \frac{\mathbf{p}^2}{2\tilde{m}_r} + Fr - \tilde{a} + \tilde{c}\sigma_1\sigma_2, \quad (10)$$

with the abbreviations

$$\tilde{c} = \frac{\beta}{3} \left\langle \frac{1}{r} \right\rangle = \frac{F}{3m_1m_2} \left\langle \frac{1}{r} \right\rangle, \quad (11)$$

$$\tilde{a} = a + \left\langle \frac{1}{r} \right\rangle \frac{\beta}{3} = a + \frac{3\tilde{c}}{2} \left(\frac{m_1}{m_2} + \frac{m_2}{m_1} \right), \quad (12)$$

$$\frac{m_r}{\tilde{m}_r} = 1 + \frac{(F\langle r \rangle - a)}{8(m_1 + m_2)} \left[16 - \frac{m_1}{m_2} - \frac{m_2}{m_1} \right]. \quad (13)$$

Its eigenvalues are

$$E_n = -\tilde{a} + \omega \xi_0 + \omega \eta_n + \tilde{c}\sigma_1\sigma_2, \quad \omega = \left[\frac{F^2}{2\tilde{m}_r} \right]^{\frac{1}{3}}, \quad (14)$$

with the negative zeros of the Airy functions ξ_n and $\eta_n = \xi_n - \xi_0$. A few ones are tabulated in App. A. The invariant mass squares

$$M_n^2 = (m_1 + m_2)^2 + 2(m_1 + m_2)(-\tilde{a} + \xi_0\omega + \eta_n\omega + \tilde{c}\sigma_1\sigma_2), \quad (15)$$

are then related to experiment.

For equal masses $m_1 = m_2 = m$, the model has the 3 parameters m , F and a . One thus needs 3 empirical data to determine them. I choose:

$$\begin{aligned} M_{d\bar{u},t1}^2 &= 4m^2 + 4m(-\tilde{a} + \xi_0\omega + \tilde{c} + \eta_1\omega), \\ M_{d\bar{u},t0}^2 &= 4m^2 + 4m(-\tilde{a} + \xi_0\omega + \tilde{c}), \\ M_{d\bar{u},s0}^2 &= 4m^2 + 4m(-\tilde{a} + \xi_0\omega - 3\tilde{c}). \end{aligned} \quad (16)$$

The spectrum is labeled self explanatory by the flavor composition $M_n = M_{d\bar{u},tn}$ or $M_n = M_{d\bar{u},sn}$, for singlets or triplets, respectively. The triple chosen in Eq.(16) exposes a certain asymmetry. The excited ρ is chosen since its experimental limit of error is very much smaller than the one for the corresponding π state. Only its ground state mass is known very accurately, *i.e.* $m_{\pi^+} = 139.57018 \pm$

Table 2. Dependence on the fudge factors.

f_r^*	f_i^*	$m_{d,u}$	m/\tilde{m}	a	$10F$	4^1S_0	4^3S_1
1	4	1.891	0.216	2.050	1.191	1.9936	2.1320
4	4	0.550	1.894	1.097	1.385	1.9936	2.1320
1	1	0.816	0.390	1.185	2.056	1.9936	2.1320
2	1	0.426	1.315	1.132	2.145	1.9936	2.1320
4	1	0.215	5.018	1.611	2.171	1.9936	2.1320
10	1	0.086	30.95	3.563	2.178	1.9936	2.1320
20	1	0.043	123.5	6.992	2.179	1.9936	2.1320
40	1	0.022	493.9	13.918	2.179	1.9936	2.1320

0.00035 MeV. In the present work only the first 4 digits are used. For equal masses, the above abbreviations become

$$\begin{aligned} \tilde{c} &= \frac{F}{3m^2} \left\langle \frac{1}{r} \right\rangle, \\ \tilde{a} &= a + 3\tilde{c}, \\ \frac{m_r}{\tilde{m}_r} &= 1 + \frac{7(F\langle r \rangle - a)}{8m}. \end{aligned} \quad (17)$$

The experiment defines 2 certainly positive differences:

$$\begin{aligned} X^2 &= M_{d\bar{u},t1}^2 - M_{d\bar{u},t0}^2 = 4m \eta_1\omega, \\ Y^2 &= M_{d\bar{u},t0}^2 - M_{d\bar{u},s0}^2 = 4m 4\tilde{c}. \end{aligned} \quad (18)$$

A third one will be constructed by the observation that $\frac{\xi_0}{\eta_1} X^2 - \frac{3}{2} Y^2 - M_{d\bar{u},s0}^2 = 4ma - 4m^2$. Keeping in mind that $\omega^3 = F^2/\tilde{m}$, one can remove trivial kinematic factors and define 3 experimental quantities B , C and D by

$$\begin{aligned} B^2 &= \frac{1}{4} \left(\frac{\xi_0}{\eta_1} X^2 - \frac{3}{2} Y^2 - M_{d\bar{u},s0}^2 \right) = ma - m^2, \\ C &= \frac{3}{16 \left(\frac{1}{r} \right)} Y^2 = \frac{F}{m}, \\ D^4 &= \frac{2}{3Y^2\eta_1^3} \left\langle \frac{1}{r} \right\rangle (X^2)^3 = 8m^2 + 7mF\langle r \rangle - 7ma. \end{aligned} \quad (19)$$

Substituting $F = mC$ and $ma = B^2 + m^2$ gives

$$D^4 = [1 + 7\langle r \rangle C] (m^2)^2 - 7B^2 m^2,$$

a quadratic equation with the solution

$$m^2 = \frac{7B^2}{2[1 + 7\langle r \rangle C]} \left[1 + \sqrt{1 + \frac{4D^4}{49B^4} [1 + 7\langle r \rangle C]} \right]. \quad (20)$$

Having m , the F and a are then calculated from (19).

With Airy functions, moments of different powers in r are somewhat difficult to evaluate. Therefore Gaussian weights are used, which give

$$\frac{\langle r \rangle^2}{\langle r^2 \rangle} = \frac{8}{3\pi}, \quad \left\langle \frac{1}{r} \right\rangle^2 \langle r^2 \rangle = \frac{6}{\pi}. \quad (21)$$

In order to allow for corrections due to the true wave functions I introduce two fudge factors f^* according to

$$\begin{aligned} \langle r \rangle &= \sqrt{\frac{8\langle r^2 \rangle_\pi}{3\pi}} f_r^*, \\ \left\langle \frac{1}{r} \right\rangle &= \frac{4}{\pi \langle r \rangle} f_i^*. \end{aligned} \quad (22)$$

Table 3. S wave spectra in GeV for light unflavored mesons.

n	1S_0 Singlets Experiment ¹	$\pi^+(ud)$ Theory	n	3S_1 Triplets Experiment ¹	$\rho^+(ud)$ Theory
1	0.1396(0)	0.1396 0.1396 ² 0.150 ³ 0.497 ⁴	1	0.7685(6)	0.7685 0.7711 ² 0.769 ³ 0.846 ⁴
2	1.300(100)	1.2550 1.2650 ² 1.300 ³ 1.326 ⁴	2	1.465(25)	1.4650 1.4650 ² 0.769 ³ 1.461 ⁴
3	1.795(10)	1.6878 1.7950 ² 1.880 ³ 1.815 ⁴	3	1.700(20) ^a	1.8493 1.9230 ² 2.000 ³ 1.916 ⁴
4	—	1.9936 2.1620 ²	4	2.150(17)	2.1320 2.2912 ²

¹Hagiwara *et al* [7], ²Zhou and Pauli [8].³Godfrey and Isgur [9], ⁴Baldicchi and Prosperi [10] (a),^aCould be a D state [11].

Since all mesons have about the same size [5], by order of magnitude, these numbers are kept universal. The fudge factors are introduced here to account, in some global fashion, for the tremendous simplification introduced by replacing Eq.(5) with (10). Some large scale variations of f_r^* and f_i^* are compiled in Table 2. The mass spectra including the ground states vary very little with the fudge factors. Any variations would show up the fastest for the high excitations. For this reason, the masses for $n = 4$ are included in the table. I do not understand this insensitivity from a mathematical or numerical point of view. The major effect of f_r^* is the ease by which one can change the quark mass. A value of $f_r^* \sim 40$ leads to the 20 MeV for the quark mass quoted in [10]. In the present model, the values $f_r^* = 2$ and $f_i^* = 1$ are taken without seeking an optimum.

In principle, one could determine the heavier quark masses analytically from the hyperfine splittings. The so obtained results are, however, not very reasonable, since experimental numbers are not sufficiently accurate. Therefore, I determine them numerically from $M_{u\bar{s},s0}$, $M_{u\bar{c},s0}$ and $M_{u\bar{b},s0}$ and compile them in Table 1. The force parameter $F \sim 1100$ MeV/fm is in line with currently used string tensions.

3 Results and Discussion

Unflavored light mesons. The results for the π - ρ system are compiled in Table 3. The experimental points are taken from Hagiwara *et al* [7]. It is no surprise that theory and experiment coincide for the π^+ , the ρ^+ and the $\rho^+(1450)$, because these data have been used to determine the parameters. More surprising is, that the model reproduces the huge mass of the excited pion within the error limit. This solves for me a long standing puzzle: Why is it that the ground state has a mass of 140 MeV,

Table 4. S wave spectra in GeV for strange mesons.

n	Experiment ¹	Theory	n	Experiment ¹	Theory
1S_0 Singlets $K^+(u\bar{s})$			3S_1 Triplets $K^{*+}(u\bar{s})$		
1	0.493677(16)	0.4937 0.6048 ² 0.47 ³	1	0.89166(26)	0.8651 0.8917 ² 0.90 ³
2	1.460 ^a	1.3943 1.5480 ² 1.45 ³	2	1.629(27) ^b	1.5649 1.6808 ² 1.58 ³
3	1.830 ^a	1.8266 2.1040 ² 2.02 ³	3	—	1.9598 2.6242 ² 2.11 ³
4	—	2.1370	4	—	2.2520

¹Hagiwara *et al* [7], ²Zhou and Pauli [8].³Godfrey and Isgur [9].^aTo be confirmed; ^b J^P not confirmed.

while the first excitation with its 1300 MeV is different by an order of magnitude? — The answer is the usual one: Two large scales interfere destructively for the ground and constructively for the excited state. The two scales are the depth $a \sim 1100$ MeV and the string constant $F \sim 1100$ MeV/fm.

The remaining three experimental masses of the π - ρ sector agree reasonably well with the calculation. There is no confirmed datum for the second excited ρ^+ (3^3S_1). The third excited ρ^+ (4^3S_1) deviates from experiment by 18 MeV, which is almost within the limits of error. The second excited π^+ (3^1S_0) deviates from experiment by 107 MeV. The table includes also a comparison with other theoretical calculations. It includes the results from a recent oscillator model [8]. Their model is even simpler than the present one: it works with a hyperfine splitting, only, but suppresses the mechanism of a position dependent mass. Despite this, their results differ little from the present ones. I have included also the results from the pioneering work of Godfrey and Isgur [9] as a prototype of a phenomenological model, and from a recent advanced calculation by Baldicchi and Prosperi [10]. Note that either of these models have not much in common with the present one. The present model gives a good description for the pion, particularly the small mass of the physical pion is reproduced. All other potential models, including even Baldicchi and Prosperi, have the wellknown difficulties with that. I would have loved to compare also with Lattice Gauge Calculations but of course no data are available for excited states, particularly not for such light systems as the pion and the rho. Note that the effort on a pocket calculator is ridiculously small as compared to gigaflops years of calculations.

Strange mesons. The S wave K^+ and K^{*+} spectra are given in Table 4. The mass of the ground state of K^+ is used to determine the mass parameter m_s . The excitations for the K (n^1S_0) differ by only 60 and 3 MeV, respectively, and the spectrum for the K^* (n^3S_1) by 27 and 64 MeV. Possibly, this could even be improved by playing with the fudge parameters, but in view of the experimental situation, it is not done here. Except the ground states, the

Table 5. Ground state masses in GeV for heavy mesons.

n	Experiment ¹	Theory	n	Experiment ¹	Theory
¹ S ₀ Singlet $\bar{D}^0(u\bar{c})$			³ S ₁ Triplet $\bar{D}^{*0}(u\bar{c})$		
1	1.8645(5)	1.8645 1.9224 ² 1.88 ³	1	2.0067(5)	1.9568 2.0067 ² 2.04 ³
¹ S ₀ Singlet $B^+(ub)$			³ S ₁ Triplet $B^{*+}(ub)$		
1	5.2790(5)	5.2790 5.2965 ² 5.31 ³	1	5.3250(6)	5.3082 5.3250 ² 5.37 ³
¹ S ₀ Singlet $D_s^-(s\bar{c})$			³ S ₁ Triplet $D_s^{*-}(s\bar{c})$		
1	1.9685(6)	1.9961 2.0201 ² 1.98 ³	1	2.1124(7)	2.0665 2.0655 ² 2.13 ³
¹ S ₀ Singlet $B_s^0(sb)$			³ S ₁ Triplet $B_s^{*0}(sb)$		
1	5.3696(24)	5.3961 5.3739 ² 5.35 ³	1	5.4166(35)	5.4185 5.3885 ² 5.45 ³
¹ S ₀ Singlet $B_c^+(cb)$			³ S ₁ Triplet $B_c^{*+}(cb)$		
1	6.4(4)	6.4914 — 6.27 ³	1	—	6.4984 6.3458 ² 6.34 ³

¹Hagiwara *etal* [7], ²Zhou and Pauli [8].³Godfrey and Isgur [9],^aTo be confirmed; ^b J^P not confirmed.**Table 6.** The predicted S spectrum in GeV for heavy mesons.

ⁿ 1S ₀	$\bar{D}^0(u\bar{c})$	ⁿ 3S ₁	$\bar{D}^{*0}(u\bar{c})$
1	1.8645	1	1.9568
2	2.5997	2	2.6666
3	3.0733	3	3.1301
4	3.4380	4	3.4889
ⁿ 1S ₀	$B^+(ub)$	ⁿ 3S ₁	$B^{*+}(ub)$
1	5.2790	1	5.3082
2	5.9653	2	5.9911
3	6.4733	3	6.4971
4	6.8911	4	6.9134
ⁿ 1S ₀	$D_s^-(s\bar{c})$	ⁿ 3S ₁	$D_s^{*-}(s\bar{c})$
1	1.9961	1	2.0665
2	2.6878	2	2.7405
3	3.1427	3	3.1879
4	3.4958	4	3.5365
ⁿ 1S ₀	$B_s^0(sb)$	ⁿ 3S ₁	$B_s^{*0}(sb)$
1	5.3961	1	5.4185
2	6.0349	2	6.0549
3	6.5114	3	6.5299
4	6.9052	4	6.9227
ⁿ 1S ₀	$B_c^+(cb)$	ⁿ 3S ₁	$B_c^{*+}(cb)$
1	6.4914	1	6.4984
2	6.9688	2	6.9752
3	7.3365	3	7.3426
4	7.6467	4	7.6526

experiments carry many ambiguities about the quantum number assignment for K and K^* mesons. Both the first and the second excited state of K (2^1S_0 and 3^1S_0) are not confirmed. Another unconfirmed resonance with mass 1.629 ± 0.027 GeV lying between 2^1S_0 and 3^1S_0 was assigned to be a singlet K . Apparently there is no position for it in the K spectrum if it is an S wave state. However, according to its mass and the present work, it might well be the first excited state of K^* (2^1S_0).

Heavy mesons. The S wave $u\bar{c}$, $u\bar{b}$, $s\bar{c}$, $s\bar{b}$ and $c\bar{b}$ meson spectra are given in Table 5. No excitations were observed for these mesons. — The mass of the ground state of \bar{D}^0 is used to determine the mass parameter m_c . No much data are available for D and D^* mesons. The model prediction for 1^1S_0 of \bar{D}^{*0} is smaller than experiment by 50 MeV. — The mass of the ground state of \bar{B}^+ is used to determine the mass parameter m_b . The ground state of \bar{B}^{*+} agrees with the present model to within 17 MeV. — No experimental values in the $s\bar{c}$ mesons are used to determine the model parameters. The model deviates from the available ground states by 27 and 45 MeV. — No data in $s\bar{b}$ mesons are used to determine the model parameters. The model deviates from the experiment by 26 and 2 MeV, and is thus almost within the experimental errors. — The mass of the ground state of B_c^+ (1^1S_0) carries a large experimental error. Model and experiment agree. — The model prediction for the excited states are compiled in Table 6, for easy reference.

4 Conclusions

The agreement between the present simple model and the experiment is excellent, with small but significant deviations. Perfect agreement has not been the goal of the present work. There must be room for a possible improvements by the ‘true’ equation (5).

With the 4 mass parameters of the up/down, strange, charm and bottom quarks, the model has only 2 additional parameters for the linear potential. In principle, the fudge factors should be counted as parameter as well, but as seen above, the choice of the up/down mass and the fudge factors is strongly coupled. Thus, with 6 canonical parameters the model exposes a reasonably good agreement with all 21 available data points.

Note that renormalized gauge field theory has also 4+1+1 parameters: The 4 flavor quark masses, the strong coupling constant α_s , and the renormalization scale λ . Of course, they can be mapped into each other [1, 2].

Once one has determined the parameters in such a first guess, one should relax the model assumption, Eq.(9), and work with the full non local model, with a position dependent mass. For this one has to go back to the computer and perform the necessary fine tunings of the parameters.

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A Solution to the linear potential

Restricting to spherical symmetry ($l = 0$), the Schrödinger equation for $V(r) = Fr$ is

$$\begin{cases} \left[\frac{\mathbf{p}^2}{2m_r} + Fr - E_n \right] \Psi_n(r) = 0, \\ \left[-\frac{1}{2m_r} \frac{1}{r} \frac{d^2}{dr^2} r + Fr - E_n \right] \Psi_n(r) = 0, \\ v_n''(r) - 2m_r(Fr - E_n)v_n(r) = 0, \end{cases} \quad (23)$$

In the last step $v_n(r) = r\Psi_n(r)$ was substituted. With the dimensionless variable

$$\xi = [2m_r F]^{\frac{1}{3}} r - \xi_n, \quad (24)$$

equation (23) is mapped into the differential equation for the Airy function $u_n''(\xi) - \xi u_n(\xi) = 0$. Eigenvalues are obtained by boundary conditions, in this case $v_n(0) = 0$,

$$\begin{aligned} E_n &= \left[\frac{F^2}{2m_r} \right]^{\frac{1}{3}} \xi_n \equiv \hbar\omega \xi_n, \\ \Psi_n(r) &= \frac{1}{r} \text{Ai}\left(r[2m_r F]^{\frac{1}{3}} - \xi_n\right). \end{aligned} \quad (25)$$

The ξ_n are the negative zeros of the Airy functions [6]:

n	ξ_n	η_n	δ_n
0	2.33811	0	0
1	4.08795	1.74984	1.74984
2	5.52056	3.18245	1.43261
3	6.78670	4.44853	1.26614

(26)

The table includes $\eta_n = \xi_n - \xi_0$. The table includes also the spacings $\delta_n = \xi_n - \xi_{n-1}$, which vary slowly with n .